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CMB power spectrum from noncommutative spacetime

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Abstract: Very recent CMB data of WMAP offers an opportunity to test inflation models, in particular, the running of spectral index is quite new and can be used to rule out some models. We show that an noncommutative spacetime inflation model gives a good explanation of these new results. In fitting the data, we also obtain a relationship between the noncommutative parameter (string scale) and the ending time of inflation.

Keywords: Cosmology of Theories beyond the SM, Physics of the Early Universe.

Recent results from the Wilkinson Microwave Anisotropy Probe [1]–[3] give further determination of cosmological parameters and constrain properties of inflationary models tightly. Some analysis was done in [3], and in particular, the new data on the running of the spectral index of the power spectrum of CMB provide good constraints on inflationary models, and some models were analyzed in [4]. In general, models satisfying the slow roll conditions naturally predict too small running of the spectral index. In this note we analyze the model of [5] in which spacetime noncommutativity is taken into account, and find that it is rather easy to accommodate WMAP data in this model. It is quite remarkable that the two parameters in this model can be used to fit four data of the spectral index. We determine these two parameters, and the string scale is in turn determined once the termination time of inflation is known. For a typical termination time of inflation $t_* = 10^{-32}$ s [6], the string scale is $l_s = 10^{-25}$ cm, or $M_s = 10^{11}$ GeV.

Spacetime in general is noncommutative in string theory, this manifests in a general relation $\Delta X \Delta T \geq l_s^2$. However it is hard to formulate string theory in a fashion to incorporate this relation directly. On a general ground, if inflation reflects physics at a scale close to string scale or a related scale, one expects that spacetime uncertainty must have effects in the CMB power spectrum, observable or to be observable. Lacking a manifest formulation of noncommutativity, we shall be content with a scheme proposed in [5], in which spacetime noncommutativity is put in by hand for a scalar field (inflaton). Skipping the detailed discussions leading to their action, we copy the free scalar action here

$$S = V \int d\eta d^3k z_k^2(\eta)(\phi'_k \phi'_{-k} - k^2 \phi_k \phi_{-k}) \,, \tag{1}$$

where

$$\begin{aligned} z_k^2 &= a(\eta)^2 y_k^2(\eta) \,, & y_k^2 &= (\beta_k^+ \beta_k^-)^{1/2} \,, \\ \frac{d\eta}{d\tau} &= \left(\frac{\beta_k^-}{\beta_k^+}\right)^{1/2} \,, & \beta_k^\pm &= \frac{1}{2}(a^{\pm 2}(\tau + l_s^2 k) + a^{\pm 2}(\tau - l_s^2 k) \,, \end{aligned} \tag{2}$$

where l_s is the string length scale, $a(\tau)$ is the scale factor in the metric, and time τ is defined such that

$$ds^2 = -a^{-2}(\tau)d\tau^2 + a^2(\tau)dx^2 \,. \tag{3}$$

Unlike in the commuting spacetime, the all important scale factor z_k in our case depends on the comoving wave number k.

We shall consider the power-law inflation, and this can be generated by a scalar potential assuming an exponential form. Let t be the cosmological time, so $a(t) = a_0 t^n$, $n > 1$. In terms of $\tau = \frac{a_0}{n+1} t^{n+1}$, $a = \alpha_0 \tau^{\frac{n}{n+1}}$ with

$$\alpha_0 = a_0^{\frac{1}{n+1}} (n+1)^{\frac{n}{n+1}} \,. \tag{4}$$

To simplify relations that will follow, we define a scale l so that $a(\tau) = (\tau/l)^{\frac{n}{n+1}}$ and $a(t) = (\frac{t}{(n+1)l})^n$. We shall estimate the relationship between the scale l and the termination time t_* later.

The scalar power spectrum can be computed in the usual way, and is given by

$$P(k) = \frac{k^2}{4\pi^2 z_k^2(\eta_k)}\,, \tag{5}$$

and η_k is determined by the condition

$$k^2 = \frac{z_k''}{z_k}\,. \tag{6}$$

Two extremal limits were considered in [5], namely the UV limit and the IR limit. However, the observed results must lie in between, so we need to calculate the power spectrum again. The interested region is actually UV in terms of the time when the perturbation is created, but this was not discussed in [5]. What we need to compute is

$$n_s = 1 + \frac{dP(k)}{d\ln k}, \quad \frac{dn_s}{d\ln k}\,. \tag{7}$$

Define

$$k_c = \left(\frac{(n+1)^2}{n(2n-1)}\right)^{-\frac{n+1}{4}} \left(\frac{l_s}{l}\right)^{n-1} l^{-1}\,. \tag{8}$$

Naively, k_c is determined by two microscopic scales l and l_s, one would expect that k_c is also at the microscopic scale. This is wrong, due to the large exponent n. To achieve sufficient number of e-folds, n ought to be large enough, and indeed we will determine it be around 13. Thus, if l_s is smaller than l by a few order, the characteristic wave length $1/k_c$ is macroscopic. The interesting range of k is between $(0.002, 0.05)\,\mathrm{Mpc}^{-1}$, and we assume these length scales be smaller than $1/k_c$, so $k_c/k \ll 1$. In solving the crossing horizon condition (6), we will expand everything in k_c/k.

Without the string spacetime uncertainty, or let $l_s = 0$, the crossing horizon time is

$$\tau_k^0 = l_s^2 k \left(\frac{k}{k_c}\right)^{\frac{2}{n-1}}\,, \tag{9}$$

the dependence on l_s is fictitious, since a factor in k_c cancels l_s^2. Since string uncertainty introduces a uncertainty in time $\sigma = l_s^2 k$, this ought to be smaller than the time when the perturbation is created, namely, σ/τ_k^0 must be small. This ratio is nothing but

$$\frac{\sigma}{\tau_k^0} = \left(\frac{k_c}{k}\right)^{\frac{2}{n-1}} \ll 1\,. \tag{10}$$

For a large enough n, k_c is small enough so that the above assumption will be valid.

In solving (6), not only we need to modify z_k according to (2), but also we need to modify the definition of the conformal time according to (2). For instance

$$\frac{d}{d\eta} = \alpha_0^2 \tau^{\frac{2n}{n+1}} \left(1 - \frac{n}{n+1}\frac{\sigma^2}{\tau^2}\right) \frac{d}{d\tau}\,, \tag{11}$$

where we have truncated to the first nontrivial order in σ/τ. After a lengthy but straightforward calculation, we find

$$\frac{z_k''}{z_k} = \frac{n(2n-1)}{(n+1)^2}\alpha_0^4\tau^{\frac{2(n-1)}{n+1}}\left(1 - \frac{2n(n+1)(2n-5)}{(2n-1)(n+1)^2}\frac{\sigma^2}{\tau^2}\right). \tag{12}$$

Solving (6) using (12), we obtain

$$\tau_k = l_s^2 k\left(\frac{k}{k_c}\right)^{\frac{2}{n-1}}\left(1 + \frac{n(2n-5)}{(n-1)(2n-1)}\left(\frac{k_c}{k}\right)^{\frac{4}{n-1}}\right). \tag{13}$$

The power spectrum is readily calculated to be

$$P(k) \sim k^{-\frac{2}{n-1}}\left(1 - \frac{4n^2(n-2)(2n+1)}{(n+1)^2(n-1)(2n-1)}\left(\frac{k_c}{k}\right)^{\frac{4}{n-1}}\right). \tag{14}$$

Denote the coefficient in the front of $(k_c/k)^{\frac{4}{n-1}}$ in the above formula by x (the positive one), we compute the spectral index and the running of the index

$$\begin{aligned} n_s - 1 &= -\frac{2}{n-1} + \frac{4x}{n-1}\left(\frac{k_c}{k}\right)^{\frac{4}{n-1}}, \\ \frac{dn_s}{d\ln k} &= -\frac{16x}{(n-1)^2}\left(\frac{k_c}{k}\right)^{\frac{4}{n-1}}. \end{aligned} \tag{15}$$

Since x is positive, n_s is larger for smaller k, the expected behavior. Now, use the WMAP data

$$\begin{aligned} n_s &= 0.93 \pm 0.03\,, & \frac{dn_s}{d\ln k} &= -0.031^{+0.016}_{-0.017} & \text{at}\quad k &= 0.05\,\mathrm{Mpc}^{-1}\,, \\ n_s &= 1.10^{+0.07}_{-0.06}\,, & \frac{dn_s}{d\ln k} &= -0.042^{+0.021}_{-0.020} & \text{at}\quad k &= 0.002\,\mathrm{Mpc}^{-1} \end{aligned} \tag{16}$$

to see whether the main formulas in (15) can fit these. There are only two parameters in (15), namely n and k_c, but there are four independent results from the WMAP. We will adopt the following strategy: we first use the two results at $k = 0.05\,\mathrm{Mpc}^{-1}$ to determine the two parameters, and make a prediction about n_s and its running at $k = 0.002\,\mathrm{Mpc}^{-1}$. Later, we use the results about n_s at two different scales to make a prediction about the running of n_s at the two scales.

Using the results at $k = 0.05\,\mathrm{Mpc}^{-1}$, we find

$$n = 13.171^{+6.198}_{-2.772}\,, \qquad k_c = 1.65^{+6.84}_{-1.64} \times 10^{-5}\,\mathrm{Mpc}^{-1}\,. \tag{17}$$

Indeed, k_c/k is a small number, and $(k_c/k)^{\frac{4}{n-1}}$ is also small enough so the approximation used in deriving (15) is valid. These parameters together with (15) predict

$$n_s = 1.11^{+0.14}_{-0.13}\,, \qquad \frac{dn_s}{d\ln k} = -0.089^{+0.059}_{-0.100} \quad \text{at} \quad k = 0.002\,\mathrm{Mpc}^{-1}\,. \tag{18}$$

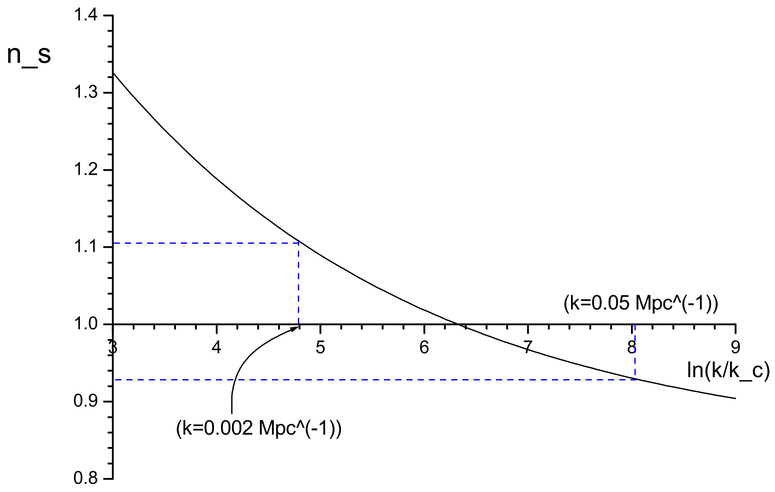

Figure 1: n_s as a function of $\ln k$. Parameters are determined using data at $k = 0.05\,\mathrm{Mpc}^{-1}$.

The central value of the predicted n_s is quite good compared to that in (16), the central value of the predicted running is larger than that in (16), but within the error bar, we notice that the tendency of the predicted running is good, it is larger at a larger scale. For $k = 0.002\,\mathrm{Mpc}^{-1}$, it is possible that higher orders in k_c/k are not negligible, and may improve our result.

Next, using the values of n_s at two different scales in (16), we determine

$$n = 13.336^{+5.204}_{-2.412}\,, \qquad k_c = 1.43^{+3.81}_{-1.42} \times 10^{-5}\,\mathrm{Mpc}^{-1}\,, \tag{19}$$

and the running of the spectral index

$$\frac{dn_s}{d\ln k}(k = 0.05\,\mathrm{Mpc}^{-1}) = 0.030^{+0.013}_{-0.011}\,, \qquad \frac{dn_s}{d\ln k}(k = 0.002\,\mathrm{Mpc}^{-1}) = -0.085^{+0.050}_{-0.065}\,. \tag{20}$$

We show the spectral index and the running as functions of $\ln k$ in figure 1 and figure 2.

We have used the WMAP data to determine two parameters n and k_c in the noncommutative inflation model. Next, we want to determine the relation between the ending time of inflation and the string scale. To do so, note that the comoving wave numbers are measured at the present time, so the normalization of the scale factor $a(t)$ at the present t_0 is 1. Since the termination of inflation, our universe underwent a radiation dominated epoch and matter dominated epoch (for simplicity we ignore the observed dark energy), so the scale factor was enhanced twice:

$$a(t_*)\left(\frac{10^4 \times 365 \times 24 \times 60^2}{t_*}\right)^{1/2}\left(\frac{13.7 \times 10^9}{10^4}\right)^{2/3} = 1\,, \tag{21}$$

where t_* is the termination time of inflation and its unit is second. Using $a(t_*) = a_0 t_*^n$, we find the parameter l

$$l = \frac{t_*}{n+1}\left(\frac{4.8 \times 10^9}{t_*}\right)^{\frac{1}{2n}}\,. \tag{22}$$

JHEP06(2003)014

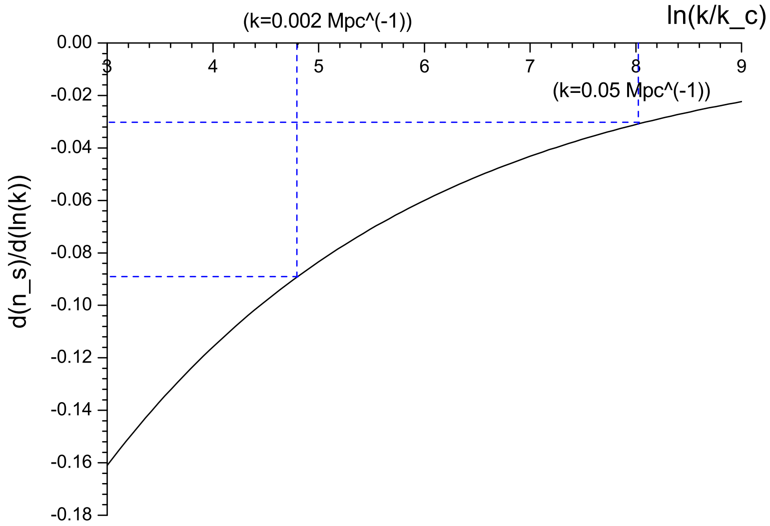

Figure 2: $dn_s/d\ln k$ as a function of $\ln k$. Parameters are determined using data at $k = 0.05\,\mathrm{Mpc}^{-1}$.

l is larger than t_*. Next, using the definition of k_c in (8), we get

$$l_s = \left[k_c l^n \left(\frac{(n+1)^2}{n(2n-1)} \right)^{\frac{n+1}{4}} \right]^{\frac{1}{n-1}} . \tag{23}$$

In the end, l and l_s are expressed in terms of t_*

$$l = dt_*^{\frac{2n-1}{2n}} , \qquad l_s = ct_*^{\frac{2n-1}{2n-2}} , \tag{24}$$

where in defining d and c, we have turned l and l_s in the unit cm.

For parameters in (17), we have

$$d = 1.18^{+1.15}_{-0.71} \times 10^{10} , \qquad c = 2.70^{+0.15}_{-0.35} \times 10^{8} . \tag{25}$$

Thus for $t_* = 10^{-32}$s, $l \sim 10^{-21}$ cm, $l_s \sim 10^{-25}$ cm. l_s is larger than the Planck scale by 8 orders. For $t_* = 10^{-38}$s, the earliest possibility [6], $l_s \sim 10^{-31}$ cm, larger than the Planck scale only by two orders. For parameters in (19),

$$d = 1.14^{+0.86}_{-0.62} \times 10^{10} , \qquad c = 2.70^{+0.21}_{-0.31} \times 10^{8} , \tag{26}$$

not much different from those in (25).

To summarize, we have found that the noncommutative inflation model accommodates the recent WMAP data nicely, and if this model possesses any truth about string theory, string theory can already be tested in the observational cosmology, this is very exciting. Although the WMAP results on the spectral index and its running are obtained by combining other experiments and are not yet very conclusive, we believe that future refined results will offer better opportunity for testing whether spacetime uncertainty is a viable physical model for inflation.

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JHEP06(2003)014